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Packing densities of more 2-block patterns

Dan Warren

Department of Mathematics, University of Florida, Gainesville FL 32611, USA

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Abstract

In this paper, we generalize the inductive techniques used in a previous paper [D. Warren, Optimal packing behavior of some 2-block patterns, Ann. Combin. 8 (2) (2004) 355–367] to prove a general result by which we may calculate the packing density of a 2-block pattern given the packing behavior of a smaller pattern having the same ratio of block sizes. We apply this result to compute the packing densities of patterns having layer sequences of the form $(1^{\alpha}, \beta)$ in the case that $\beta \geqslant \alpha \geqslant 2$ and $\alpha \mid 2\beta$. Necessary unimodality and maximality results are proven as we need them.

MSC: 05A15; 05A16

Keywords: Pattern containment; Permutations; Layered permutations; Packing density

1. Preliminaries

Let $m \leqslant n$ be positive integers, and let $\pi \in S_m$ and $\sigma \in S_n$ be permutations. We say an m-subset S of $[n]$ is an *occurrence* of π in σ if the restriction of σ to S is isomorphic to π as a linear order, that is, the elements of $\sigma\lceil_S$ are in the same relative order as those of π. We say σ avoids π if it has no occurrences of π.

The problem of characterizing the n-permutations that avoid a certain subpattern is well known and has long been the subject of a large body of research (in fact, entire two chapters of [2] are devoted to issues related to avoidance). In 1992, at a SIAM meeting on Discrete Mathematics, an opposite question of sorts was posed by Herb Wilf: what if, instead of trying to characterize permutations that avoid a certain pattern, we look at permutations that have a maximal number of occurrences of a pattern? The subject of permutation packing was born.

E-mail address: warren@math.ufl.edu.

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The largest body of work on the subject of permutation packing is the PhD thesis [4] of Alkes Price, whose notation we will adopt where we can. For $\pi \in S_m$ and $\sigma \in S_n$, let $g(\pi, \sigma)$ be the number of occurrences of π in σ. For each $n \in \mathbb{Z}^+$, let

$$g(\pi, n) := \max_{\sigma \in S_n} g(\pi, \sigma).$$

If $\sigma \in S_n$ and $g(\pi, \sigma) = g(\pi, n)$, we say σ is π-optimal over S_n. That $g(\pi, n) \leqslant \binom{n}{m}$ for every $\pi \in S_m$ is clear, since every occurrence of π in σ is by definition an m-subset of $[n]$. It was conjectured by Wilf that $g(\pi, n)$ is asymptotically proportional to $\binom{n}{m}$, and the following stronger result was later proven by Galvin:

Lemma 1.1 (Galvin). *The sequence* $\bigl(g(\pi, n)/\binom{n}{m}\bigr)_{n \geqslant m}$ *is nonincreasing in* n.

In particular, a limit exists. We define the *packing density of* π to be

$$\rho(\pi) := \lim_n \frac{g(\pi, n)}{\binom{n}{m}}.$$

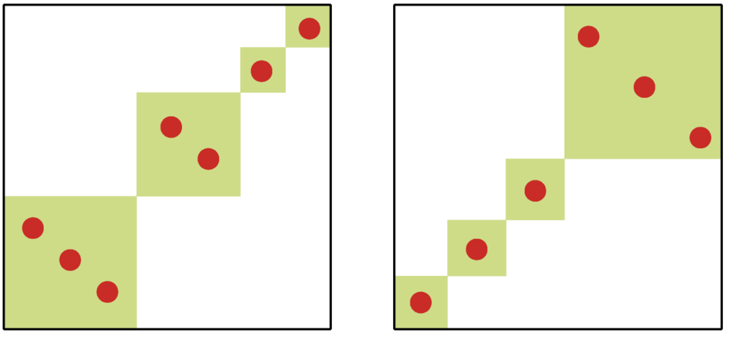
In many cases, the packing density of a permutation may be determined by asymptotic means without ever proving any finite structure is optimal. For example, a technique pioneered by Price and used by Peter Hästö in [3] is to characterize the behavior of large permutations with a fixed, small number of layers using partitions of unity. In this technique, the packing density of some patterns may be computed by maximizing the value of a polynomial in several variables instead of actually counting occurrences in specific permutations. One particularly direct way $\rho(\pi)$ is computed is to find an infinite family (σ_k) of permutations $\sigma_k \in S_{n_k}$ that are π-optimal and then compute $\lim_n g(\pi, \sigma_k)/\binom{n_k}{m}$ by ordinary means. In this case the object of the packing density problem extends to the problem of characterizing the structure of their optimal permutations.

The structure most often imposed on patterns in order to determine their packing density is that of layering. A pattern π is called layered if it can be decomposed into an array of descending sequences of consecutive integers, which are then ordered increasingly by first elements. Alternately, layered permutations are those which have no descents of size greater than 1. Notice that a layered permutation is uniquely determined by its ordered list of layer lengths (its *type*); the layered pattern of type $(3, 2, 1, 1)$ is shown in Fig. 1(a). An early result due to Stromquist, proven first in [5] for posets and then reproduced in [4] for permutations, gives us the distinct advantage that when π is layered, we are able to restrict our search for a π-optimal permutation to the (much smaller) class of layered permutations on n:

Theorem 1.2 (Stromquist). *Let* $\pi \in S_m$ *be layered. Then,*

$$\begin{aligned} g(\pi, n) &= \max\bigl\{g(\pi, \sigma) :: \sigma \in S_n\bigr\} \\ &= \max\bigl\{g(\pi, \sigma) :: \sigma \in S_n \textit{ is layered}\bigr\}. \end{aligned}$$

In general, the lack of this restriction is a large part of what makes the problem of finding packing densities of unlayered patterns so difficult. The problem we next encounter when computing packing densities of layered patterns is that even in the case that π is layered, the number of layers in a π-optimal permutation need not be bounded; the issue of when this problem occurs



(a) (b)

Fig. 1. (a) 3215467 is the unique layered permutation of type $(3, 2, 1, 1)$. (b) The structure of $\tau_{\langle\alpha,\beta\rangle}$ is shown for $\alpha = \beta = 3$. Layers in both are shown in shaded boxes.

was dealt with extensively in [3]. Such is the case with the particular class of patterns with which this paper deals. We will use the same notation as [6] to simplify the text of the proofs:

Notation. Let $\tau_{\langle\alpha,\beta\rangle}$ denote the layered pattern of type $(1^\alpha, \beta)$. $\tau_{\langle 3,3\rangle}$ is shown in Fig. 1(b).

In [6], an *antilayer*[1] was defined to be a sequence of consecutive layers of size 1, and this concept was applied to deal with the apparent phenomenon that patterns with long sequences of layers of size 1 tend to have optimal permutations with long sequences of layers of size 1, nicely complementing the fact that patterns with large layers tend to have optimal permutations with large layers. In our context, a 2-block pattern is just one of the form $\tau_{\langle\alpha,\beta\rangle}$ for $\alpha, \beta \geqslant 2$. α and β are called the block sizes.

2. A more careful look at $\tau_{\langle\alpha,\beta\rangle}$

In [1], the structure of a $\tau_{\langle 2,2\rangle}$-optimal permutation of size n was explicitly characterized, and in [6], it was proven by an inductive technique that the $\tau_{\langle\alpha,\alpha\rangle}$-optimal permutation of size $2n$ always had exactly the same structure, independent of α, namely a single antilayer followed by a single layer of the same length. In general, if it can be proven that there is a $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation of size n having a 2-block structure (a single antilayer followed by a single layer), then the maximum number of occurrences of $\tau_{\langle\alpha,\beta\rangle}$ in a permutation of length n is

$$\max_{0\leqslant k\leqslant n} \binom{k}{\alpha}\binom{n-k}{\beta}. \tag{1}$$

In [6], the existence of a 2-block $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation on $[n]$ was proven in the case that $\alpha = 2$ and $\beta \geqslant 3$, for n divisible by $\beta + 2$. Although we were then able to compute the packing density of $\tau_{\langle 2,\beta\rangle}$ via asymptotic methods, we would like a stronger result characterizing the optimal layout of permutations of all lengths, that is, we would like to know the value of k achieving (1). We make use of the following result.

[1] To justify the terminology, layers and antilayers correspond to antichains and chains, respectively, when translated into the language of posets.

Proposition 2.1. *Let $\alpha, \beta \in \mathbb{Z}^+$. The function*

$$B(x) := \binom{x}{\alpha}\binom{n-x}{\beta} \tag{2}$$

$(x \in \mathbb{R})$ *is unimodal on $[\alpha, n-\beta]$ and achieves its maximum value on that interval at some point x_0 in the interval $\left[\frac{\alpha n-\beta}{\alpha+\beta}, \frac{\alpha n+\alpha}{\alpha+\beta}\right)$.*

Proof. To begin with, we may see that $B(x)$ is log-concave by simply checking that the second derivative of $\ln(B(x))$ is always negative for $x \in [\alpha, n-\beta]$, so that $\ln(B(x))$ is always concave down on that interval. Unimodality of $B(x)$ follows immediately. It remains to pinpoint the location of the maximum. By the definition of $B(x)$, we have

$$\begin{aligned}\frac{B(x)}{B(x+1)} &= \frac{(x)_\alpha}{(x+1)_\alpha}\frac{(n-x)_\beta}{(n-x-1)_\beta}\\ &= \frac{(x)_{\alpha-1}(x-\alpha+1)}{(x+1)(x)_{\alpha-1}}\frac{(n-x)(n-x-1)_{\beta-1}}{(n-x-1)_{\beta-1}(n-x-\beta)}\\ &= \frac{(x-\alpha+1)(n-x)}{(x+1)(n-x-\beta)}.\end{aligned}$$

It then follows that $B(x) \geqslant B(x+1)$ if and only if $(x-\alpha+1)(n-x) \geqslant (x+1)(n-x-\beta)$, which, after some rearrangement of terms, turns out to happen exactly when $x \geqslant \frac{n\alpha-\beta}{\alpha+\beta}$. Thus the chord on the curve $y = B(x)$ from $(x, B(x))$ to $(x+1, B(x+1))$ changes from having positive slope to having negative slope when x passes $\frac{n\alpha-\beta}{\alpha+\beta}$, so $B(x)$ must change from increasing to decreasing somewhere inside the interval $\left[\frac{\alpha n-\beta}{\alpha+\beta}, \frac{\alpha n+\alpha}{\alpha+\beta}\right)$. Since B is unimodal, the maximum value of B must be achieved in this interval. □

Corollary 2.2. *Let $n \in \mathbb{N}$ and suppose that we know there is a $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation of size n consisting of a single antilayer followed by a single layer. Then,*

$$g(\tau_{\langle\alpha,\beta\rangle}, n) \leqslant \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{\beta}.$$

If $(\alpha+\beta) \mid n$, then

$$g(\tau_{\langle\alpha,\beta\rangle}, n) = \binom{\frac{\alpha n}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{\beta}.$$

Proof. Since there is a $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation of size n of the given structure, we know that $g(\tau_{\langle\alpha,\beta\rangle}, n)$ is equal to the maximum value of $B(x)$ for $x \in \mathbb{Z}$. We know from Proposition 2.1 that the function $B(x)$ is unimodal and has its maximum somewhere in the interval $\left[\frac{\alpha n-\beta}{\alpha+\beta}, \frac{\alpha n+\alpha}{\alpha+\beta}\right)$, so the maximum value of $B(x)$ for $x \in \mathbb{Z}$ is of the form $\binom{\frac{\alpha n+\xi}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n-\xi}{\alpha+\beta}}{\beta}$ for some number $-\beta \leqslant \xi \leqslant \alpha$. The first inequality then follows from the fact that the binomial function $\binom{x}{m}$ is an increasing function of x. If $(\alpha+\beta) \mid n$, the only integer in $\left[\frac{\alpha n-\beta}{\alpha+\beta}, \frac{\alpha n+\alpha}{\alpha+\beta}\right)$ is $x_0 = \frac{\alpha n}{\alpha+\beta}$. It now follows that the maximum value of $B(x)$ for $x \in \mathbb{Z}$ is $B(x_0)$, hence $g(\tau_{\langle\alpha,\beta\rangle}, n) = B(x_0) = \binom{\frac{\alpha n}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{\beta}$. □

3. Application to larger patterns

We may now begin to reapply the inductive ideas applied to the pattern $\tau_{\langle\alpha,\alpha\rangle}$ in [6], using more general 2-block patterns as base cases. So that the induction flows smoothly, we will first need to prove the following technical lemma.

Lemma 3.1. *Let $a, b, n, r \in \mathbb{Z}^+$ and suppose that $r \leqslant a$, $n \geqslant a + b$, and $a \mid rb$. Let $z := \frac{rb}{a}$, and let*

$$f := \frac{n}{a+b} \quad \textit{and} \quad g := \frac{n-(z+r)}{(a+b)-(z+r)}.$$

Then,

$$af - r = (a-r)g$$

and we have the following equality among generalized binomial coefficients:

$$\binom{a}{r}\binom{af}{a} = \binom{af}{r}\binom{(a-r)g}{a-r}. \tag{3}$$

Proof. That $af - r = (a-r)g$ can be determined by elementary algebraic manipulation. The equality (3) on integer binomial coefficients then has the following easy proof: choosing an a-set A from a set S of size af and then choosing an r-subset from A is the same as first choosing an r-subset from S, then choosing the remaining $a-r$ elements from the remaining $(a-r)g$. Since these two functions are polynomials, and they agree on an infinite number of points, they must agree everywhere. □

We can now move on to our first main result.

Theorem 3.2. *Let $\alpha, \beta, n \in \mathbb{N}$, and suppose $(\alpha+\beta) \mid n$. Suppose that there is known to be a $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation of size n consisting of a single antilayer followed by a single layer. Then, for every $k \in \mathbb{Z}^+$, we have*

$$g(\tau_{\langle k\alpha,k\beta\rangle}, n) = \binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta}.$$

Proof. Since $(\alpha+\beta) \mid n$, that $g(\tau_{\langle k\alpha,k\beta\rangle}, n) \geqslant \binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta}$ follows from constructing a permutation on $[n]$ consisting of a single antilayer of size $\frac{\alpha n}{\alpha+\beta}$ and a single layer of size $\frac{\beta n}{\alpha+\beta}$. To prove the reverse inequality, we will proceed by induction on k. By Corollary 2.2, our base case $k = 1$ is covered, that is, we know

$$g(\tau_{\langle\alpha,\beta\rangle}, n) = \binom{\frac{\alpha n}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{\beta}.$$

Assume that the theorem holds for $1, \dots, k-1$. Suppose that

$$g(\tau_{\langle k\alpha,k\beta\rangle}, n) \geqslant \binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta} + \delta$$

for some $\delta > 0$, and let $\sigma \in S_n$ be a $\tau_{\langle k\alpha,k\beta\rangle}$-optimal permutation in S_n. In each occurrence of $\tau_{\langle k\alpha,k\beta\rangle}$ in σ, there are $\binom{k\alpha}{\alpha}\binom{k\beta}{\beta}$ occurrences of $\tau_{\langle\alpha,\beta\rangle}$. Consider an occurrence τ_0 of $\tau_{\langle\alpha,\beta\rangle}$ in σ and occurrence τ^* of $\tau_{\langle k\alpha,k\beta\rangle}$ containing it. The elements of $\tau^* \setminus \tau_0$ must form an occurrence of $\tau_{\langle(k-1)\alpha,(k-1)\beta\rangle}$ in the remaining $n-(\alpha+\beta)$ elements of σ, so the number of occurrences of $\tau_{\langle k\alpha,k\beta\rangle}$ in σ which can contain a single occurrence of $\tau_{\langle\alpha,\beta\rangle}$ is bounded above by $g(\tau_{\langle(k-1)\alpha,(k-1)\beta\rangle}, n-(\alpha+\beta))$. Since $(\alpha+\beta) \mid n$, of course $(\alpha+\beta) \mid (n-(\alpha+\beta))$. Hence, we may apply the induction hypothesis to get

$$g\big(\tau_{\langle(k-1)\alpha,(k-1)\beta\rangle}, n-(\alpha+\beta)\big) \leqslant \binom{\frac{\alpha(n-(\alpha+\beta))}{\alpha+\beta}}{(k-1)\alpha}\binom{\frac{\beta(n-(\alpha+\beta))}{\alpha+\beta}}{(k-1)\beta}.$$

Now, applying Lemma 3.1 (first using $\{a = k\alpha, b = k\beta, r = \alpha\}$, then $\{a = k\beta, b = k\alpha, r = \beta\}$), we have

$$\begin{aligned}
g(\tau_{\langle\alpha,\beta\rangle}, \sigma) &\geqslant \frac{\binom{k\alpha}{\alpha}\binom{k\beta}{\beta}}{g(\tau_{\langle(k-1)\alpha,(k-1)\beta\rangle}, n-(\alpha+\beta))}\left[\binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{kk\beta} + \delta\right] \\
&\geqslant \frac{\binom{k\alpha}{\alpha}\binom{k\beta}{\beta}}{\binom{\frac{\alpha(n-(\alpha+\beta))}{\alpha+\beta}}{k(k-1)\alpha}\binom{\frac{\beta(n-(\alpha+\beta))}{\alpha+\beta}}{k(k-1)\beta}}\left[\binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta} + \delta\right] \\
&= \binom{\frac{\alpha n}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{\beta} + \varepsilon
\end{aligned}$$

for some $\varepsilon > 0$, which contradicts the result of Corollary 2.2. The result follows. □

Notice that Theorem 3.2 also characterizes the structure of a $\tau_{\langle k\alpha,k\beta\rangle}$-optimal permutation in the cases where it applies.

Corollary 3.3. *Suppose $(\alpha+\beta) \mid n$. If there is a $\tau_{\langle\alpha,\beta\rangle}$-permutation $\sigma \in S_n$ consisting of a single antilayer followed by a single layer, then σ is in fact $\tau_{\langle k\alpha,k\beta\rangle}$-optimal for every $k \in \mathbb{N}$.*

If we drop the divisibility condition on n, we can still get an upper bound on $g(\tau_{\langle\alpha,\beta\rangle}, n)$ by similar means.

Theorem 3.4. *Let $\alpha, \beta, n \in \mathbb{N}$, and let x be a real number s.t. $x\alpha, x\beta \in \mathbb{N}$. Suppose there is known to be a $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation of size n consisting of a single antilayer followed by a single layer, and we know*

$$g(\tau_{\langle x\alpha,x\beta\rangle}, n) \leqslant \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{x\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{x\beta}.$$

Then for every $r \in \mathbb{N}$ we have

$$g(\tau_{\langle (x+r)\alpha,(x+r)\beta\rangle}, n) \leqslant \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{(x+r)\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{(x+r)\beta}.$$

Proof. The proof is done here by induction on r; we assume the case $r=0$ in the hypotheses. The induction step is done by a similar approach to that of Theorem 3.2. Assume the theorem holds for $0,\dots,r-1$, and suppose that

$$g(\tau_{\langle (x+r)\alpha,(x+r)\beta\rangle}, n) \geqslant \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{(x+r)\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{(x+r)\beta} + \delta$$

for some $\delta > 0$. Let σ be a $\tau_{\langle (x+r)\alpha,(x+r)\beta\rangle}$-optimal permutation on $[n]$. As before, we now count the number of occurrences of $\tau_{\langle\alpha,\beta\rangle}$ in σ. Certainly in any occurrence of $\tau_{\langle (x+r)\alpha,(x+r)\beta\rangle}$ in σ, there are $\binom{(x+r)\alpha}{\alpha}\binom{(x+r)\beta}{\beta}$ occurrences of $\tau_{\langle\alpha,\beta\rangle}$. Let τ_0 be a specific occurrence of $\tau_{\langle\alpha,\beta\rangle}$ in σ. For any occurrence τ^* of $\tau_{\langle (x+r)\alpha,(x+r)\beta\rangle}$ containing τ_0, the remaining $(x+r-1)(\alpha+\beta)$ entries of τ^* must form an occurrence of $\tau_{\langle (x+r-1)\alpha,(x+r-1)\beta\rangle}$ in the remaining $(n-(\alpha+\beta))$ entries of σ. Hence the number of occurrences of $\tau_{\langle (x+r)\alpha,(x+r)\beta\rangle}$ containing τ_0 is bounded above by $g(\tau_{\langle (x+r-1)\alpha,(x+r-1)\beta\rangle}, n-(\alpha+\beta))$. Now, by induction, we know that

$$g\big(\tau_{\langle (x+r-1)\alpha,(x+r-1)\beta\rangle}, n-(\alpha+\beta)\big) \leqslant \binom{\frac{\alpha(n-(\alpha+\beta)+\alpha}{\alpha+\beta}}{(x+r-1)\alpha}\binom{\frac{\beta(n-(\alpha+\beta))+\beta}{\alpha+\beta}}{(x+r-1)\beta},$$

so we have

$$\begin{aligned} g(\tau_{\langle\alpha,\beta\rangle},\sigma) &\geqslant \frac{\binom{(x+r)\alpha}{\alpha}\binom{(x+r)\beta}{\beta}}{g(\tau_{\langle (x+r-1)\alpha,(x+r-1)\beta\rangle}, n-(\alpha+\beta))}\left[\binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{(x+r)\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{(x+r)\beta}+\delta\right] \\ &\geqslant \frac{\binom{(x+r)\alpha}{\alpha}\binom{(x+r)\beta}{\beta}}{\binom{\frac{\alpha(n-(\alpha+\beta))+\alpha}{\alpha+\beta}}{(x+r-1)\alpha}\binom{\frac{\beta(n-(\alpha+\beta))+\beta}{\alpha+\beta}}{(x+r-1)\beta}}\left[\binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{(x+r)\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{(x+r)\beta}+\delta\right]. \end{aligned} \tag{4}$$

Since $\frac{\alpha n+\alpha}{\alpha+\beta} - \alpha = \frac{\alpha(n-(\alpha+\beta))+\alpha}{\alpha+\beta}$, a similar statement to Lemma 3.1 shows that the equality

$$\binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{(x+r)\alpha}\binom{(x+r)\alpha}{\alpha} = \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{\alpha}\binom{\frac{\alpha(n-(\alpha+\beta))+\alpha}{\alpha+\beta}}{(x+r-1)\alpha}$$

holds. Similarly, since $\frac{\beta n+\beta}{\alpha+\beta} - \beta = \frac{\beta(n-(\alpha+\beta))+\beta}{\alpha+\beta}$, we can show that

$$\binom{\frac{\beta n+\beta}{\alpha+\beta}}{(x+r)\beta}\binom{(x+r)\beta}{\beta} = \binom{\frac{\beta n+\beta}{(\alpha+\beta}}{\beta}\binom{\frac{\beta(n-(\alpha+\beta))+\beta}{\alpha+\beta}}{(x+r-1)\beta}.$$

It now follows from (4) that

$$g(\tau_{\langle\alpha,\beta\rangle},\sigma) \geqslant \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{\beta} + \varepsilon$$

for some $\varepsilon > 0$, which contradicts the results of Corollary 2.2. The result follows. □

4. Concrete results

In the case that $\alpha = 2$, the 2-block structure of a $\tau_{\langle\alpha,\beta\rangle}$-optimal permutation was proven in [6], so we may now use the results of the previous sections to compute the packing densities of a specific class of patterns. In particular, Corollary 2.2 gives us exactly the characterization we wanted of an infinite family of $\tau_{\langle 2,\beta\rangle}$-optimal permutations, which will provide an essential base case to which we can apply Theorem 4.2.

Corollary 4.1. *Suppose $\beta \geqslant 2$ and $(\beta+2) \mid n$. Then, there is a $\tau_{\langle 2,\beta\rangle}$-optimal permutation $\sigma_{2,\beta}(n)$ on $[n]$ consisting of a single antilayer of length $\frac{2n}{2+\beta}$, followed by a layer of length $\frac{\beta n}{2+\beta}$. It follows that*

$$g(\tau_{\langle 2,\beta\rangle}, n) = g(\tau_{\langle 2,\beta\rangle}, \sigma_{2,\beta}) = \binom{\frac{2n}{2+\beta}}{2}\binom{\frac{\beta n}{2+\beta}}{\beta}.$$

Theorem 4.2. *Suppose $\beta \geqslant \alpha \geqslant 2$ and $\alpha \mid 2\beta$. Then, for every n divisible by $\left(2+\frac{2\beta}{\alpha}\right)$, we have*

$$g(\tau_{\langle\alpha,\beta\rangle}, n) \leqslant \binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{\beta}.$$

Proof. By Theorem 3.4 (applied with $x = 1$ and $x = \frac{3}{2}$), we need only prove that

$$g(\tau_{\langle 2,\beta\rangle}, n) \leqslant \binom{\frac{2n+2}{2+\beta}}{2}\binom{\frac{\beta n+\beta}{2+\beta}}{\beta} \quad \text{and} \quad g(\tau_{\langle 3,\beta\rangle}, n) \leqslant \binom{\frac{3n+3}{3+\beta}}{3}\binom{\frac{\beta n+\beta}{3+\beta}}{\beta}.$$

We know that $g(\tau_{\langle 2,\beta\rangle}, n) = \binom{\frac{2n}{2+\beta}}{2}\binom{\frac{\beta n}{2+\beta}}{\beta}$ from Corollary 4.1; to prove the other inequality, we will prove that $g(\tau_{\langle 3,\beta\rangle}, n) \leqslant \binom{\frac{3n}{3+\beta}}{3}\binom{\frac{\beta n}{3+\beta}}{\beta}$.

Let $\alpha = 3$. Then, $\alpha \mid 2\beta$ means that $3 \mid \beta$. Choose n divisible by $2 + \frac{2}{3}\beta$, and let σ be a $\tau_{\langle 3,\beta\rangle}$-optimal permutation of size n. Let $r := \frac{\beta}{3}$, $y := \frac{2\beta}{\alpha} = 2r$, and suppose that

$$g(\tau_{\langle 3,\beta\rangle}, n) \geqslant \binom{\frac{3n}{3+\beta}}{3}\binom{\frac{\beta n}{3+\beta}}{\beta} + \delta$$

for some $\delta > 0$. Let $\sigma \in S_n$ be $\tau_{\langle 3,\beta\rangle}$-optimal. Certainly any occurrence of $\tau_{\langle 3,\beta\rangle}$ in σ must contain exactly $3\binom{\beta}{y}$ occurrences of $\tau_{\langle 2,y\rangle}$. Let τ_0 be an occurrence of $\tau_{\langle 2,y\rangle}$ in σ. Since $\tau_{\langle 3,\beta\rangle}$ is layered, Theorem 1.2 allows us to assume σ is layered, so let x be the location of the first element in the layer of σ that contains the layer of size y in τ_0. Now, consider an occurrence τ^* of $\tau_{\langle 3,\beta\rangle}$ that contains τ_0. It is clear that none of the first three elements of τ^* can be to the (weak) right of x and that none of the last β elements of τ^* can be to the (strict) left of x (see Fig. 2). It follows

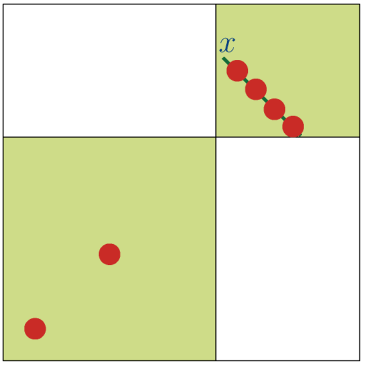


Fig. 2. An occurrence τ_0 of $\tau_{\langle 2,y\rangle}$ in an unknown permutation σ is shown for $y=4$. Notice that we can only get an occurrence of $\tau_{\langle 3,6\rangle}$ containing τ_0 by adding elements to the shaded regions—one to the left region and two to the right. The only assumption we make about σ is that it is layered.

that the number of occurrences of $\tau_{\langle 3,\beta\rangle}$ in σ that can contain τ_0 is bounded above by the number $\binom{x-3}{1}\binom{n-x-y+1}{\beta-y}$, which by Corollary 2.2[2] is bounded above by

$$\binom{\frac{n-y-2}{1+(\beta-y)}}{1}\binom{\frac{(\beta-y)(n-y-2)}{1+(\beta-y)}}{\beta-y}.$$

Since this number is independent of our choice of τ_0, it follows that no occurrence of $\tau_{\langle 2,y\rangle}$ can be contained in more than $\binom{\frac{n-y-2}{1+(\beta-y)}}{1}\binom{\frac{(\beta-y)(n-y-2)}{1+(\beta-y)}}{\beta-y}$ occurrences of $\tau_{\langle 3,\beta\rangle}$ in σ, so we have

$$\begin{aligned} g(\tau_{\langle 2,y\rangle},\sigma) &\geqslant \frac{3\binom{\beta}{y}}{\binom{\frac{n-y-2}{1+(\beta-y)}}{1}\binom{\frac{(\beta-y)(n-y-2)}{1+(\beta-y)}}{\beta-y}}\left[\binom{\frac{3n}{3+\beta}}{3}\binom{\frac{\beta n}{3+\beta}}{\beta}+\delta\right] \\ &\geqslant \binom{\frac{3n}{\beta+3}}{2}\binom{\frac{\beta n}{\beta+3}}{y}+\varepsilon \\ &= \binom{\frac{2n}{y+2}}{2}\binom{\frac{yn}{y+2}}{y}+\varepsilon \end{aligned}$$

for some $\varepsilon>0$, where the second inequality comes from Lemma 3.1 (first with $\{a=3,\ b=\beta,\ k=2\}$ and then with $\{a=\beta, b=3, k=y\}$). However, since $(2+y)\mid n$, this statement contradicts the result of Corollary 4.1 that $g(\tau_{\langle 2,y\rangle},n)=\binom{\frac{2n}{y+2}}{2}\binom{\frac{yn}{y+2}}{y}$ because by definition $g(\tau_{\langle 2,y\rangle},\sigma)$ is bounded above by $g(\tau_{\langle 2,y\rangle},n)$. The result for $\alpha=3$ follows. $\square$

The main result of this paper, the computation of the packing density, now follows as a corollary.

[2] Since we assume $(2+y)=(2r+2)\mid n$, we also know that $1+(\beta-y)=(r+1)\mid(n-2r-2)=(n-y-2)$. The reason the case $\alpha=3$ is special is because in this case, the size of $\tau^*\setminus\tau_0$ is exactly half the size of τ_0, so the divisibility works out perfectly to satisfy the divisibility hypothesis of Proposition 2.1.

Corollary 4.3. *Suppose $\beta \geqslant \alpha \geqslant 2$ and $\alpha \mid 2\beta$. Then, the packing density of the pattern $\tau_{\langle\alpha,\beta\rangle}$ is*

$$\rho(\tau_{\langle\alpha,\beta\rangle}) = \binom{\alpha+\beta}{\alpha}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}. \tag{5}$$

Proof. By letting $\sigma_{\alpha,\beta}(n)$ be the permutation on $[n]$ consisting of an antilayer of length $\lfloor\frac{\alpha n}{\alpha+\beta}\rfloor$ followed by a layer of length $\lceil\frac{\beta n}{\alpha+\beta}\rceil$, we ensure that $g(\tau_{\langle\alpha,\beta\rangle}, n) \geqslant \binom{\lfloor\frac{\alpha n}{\alpha+\beta}\rfloor}{\alpha}\binom{\lceil\frac{\beta n}{\alpha+\beta}\rceil}{\beta}$. Hence, by Theorem 4.2, we have

$$\frac{\binom{\lfloor\frac{\alpha n}{\alpha+\beta}\rfloor}{\alpha}\binom{\lceil\frac{\beta n}{\alpha+\beta}\rceil}{\beta}}{\binom{n}{\alpha+\beta}} \leqslant \frac{g(\tau_{\langle\alpha,\beta\rangle}, n)}{\binom{n}{\alpha+\beta}} \leqslant \frac{\binom{\frac{\alpha n+\alpha}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{\beta}}{\binom{n}{\alpha+\beta}}$$

for every n divisible by $2 + \frac{2\beta}{\alpha}$. Certainly the limits of our two bounds are the same, so the result (5) follows from the squeeze theorem. □

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